Programming (Econometrics)

Lecture 6: Nonlinear data structures

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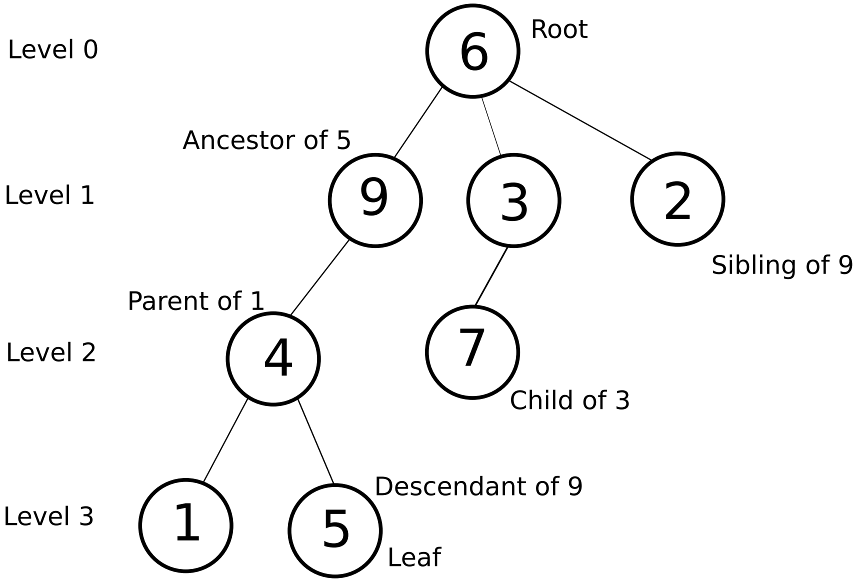
Trees


Level 0
Level 1
Level 2
Level 3
6
Root
Ancestor of 5
9
3
2
Sibling of 9
Parent of 1
4
7
Child of 3
1
5
Descendant of 9
Leaf

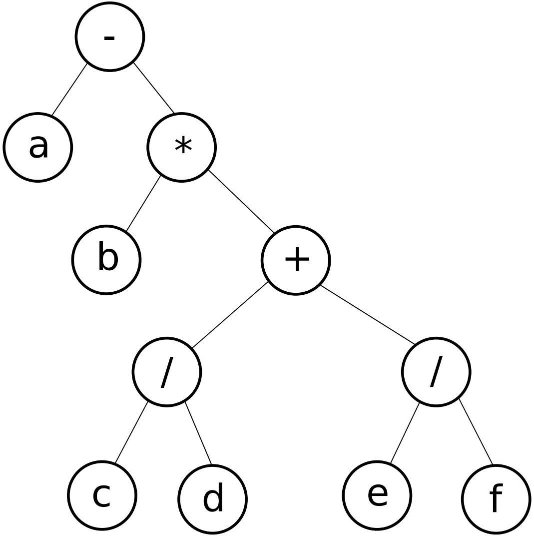
Trees: definition and implementation in Matlab

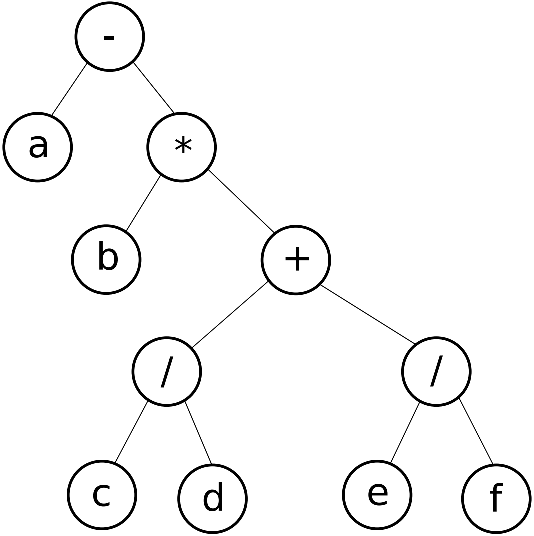
1. empty T is a tree
2. if T is not empty, a T has exactly one node designated as the root(T)
3. the remaining nodes $(T - root(T))$ of a tree are partitioned into m disjoint sets $T_1, \ldots, T_m$. Each of these are in turn a tree, and are called subtrees of T.

Tree arity m defines max amount of subtrees ($m = 1 \rightarrow$ linked list, $m = 2 \rightarrow$ binary tree)

```
classdef treeNode < handle
  properties
    key
    left
    right
  end
end
```

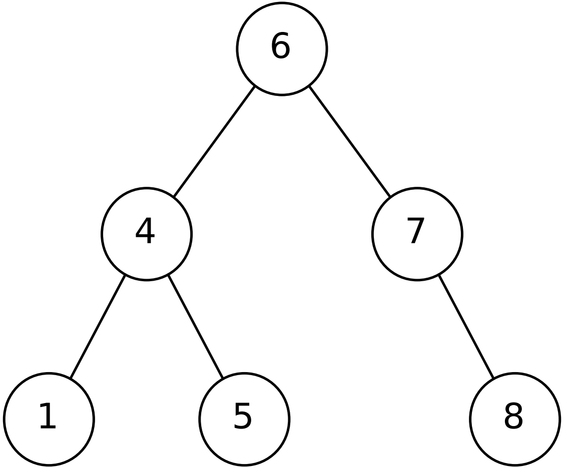
$$a - b * (c/d + e/f)$$





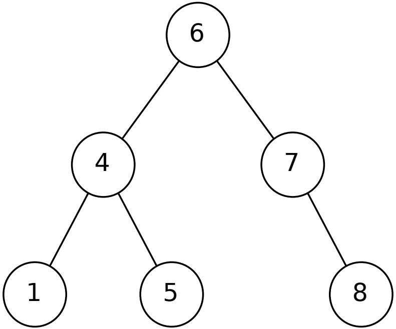
Example: tree traversal schemes; inorder, preorder and postorder

Binary search trees (BST)

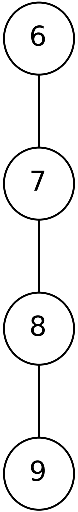


BST search time / balanced case

- Each level: 1 comparison → 1/2 remaining nodes "discarded"
- Find complexity: $O(\log_2 n)$



Extremely unbalanced tree



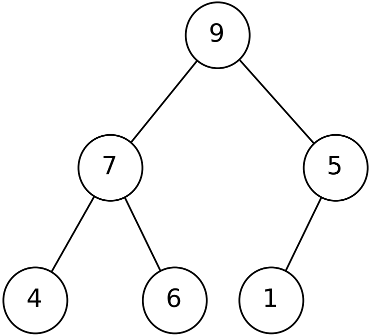
BST operation complexity

- Insert: $O(n)$ (can be lower in balanced case)
- Delete current node: $O(1)$
- Search / balanced case: $O(\log n)$
- Search / unbalanced case: $O(n)$

Heap

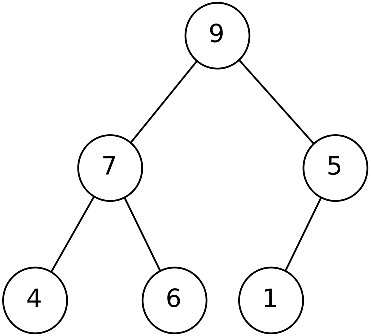
- Balanced tree: every level of depth x (except last) has exactly 2^x nodes

- Heap property: the key of each node is maximum that of its parent



Heap as an array

i^{th} element of j^{th} level is located in the index $2^j + (i-1)$



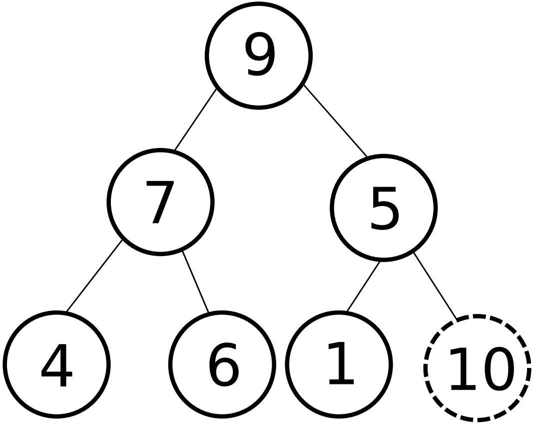
9	7	5	4	6	1

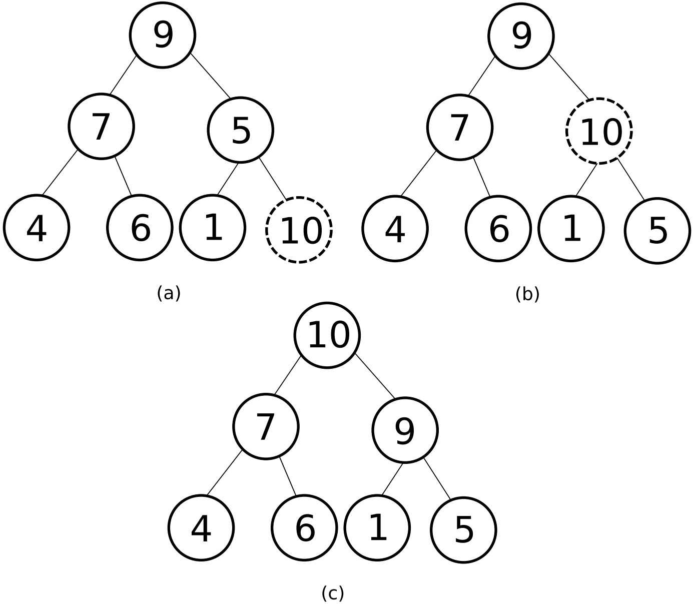
Constructing a heap

When inserting a new node, it becomes:

- Last node of the last layer, if there is space
- First node of a new layer (depth increases by 1)

$\Rightarrow$ possible violation of the heap property





Complexity?

Deleting from the heap

- *Only* the root node can be deleted

  $\implies$ *priority queue* semantics that is very useful in various cases (e.g. queueing elements that some have always priority over others)

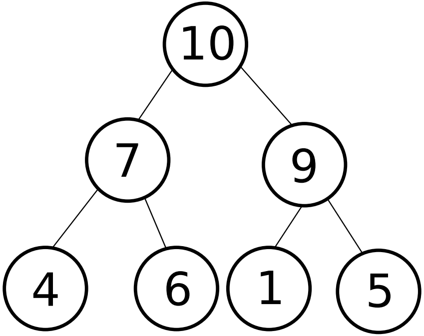
- Elegant data structure with many applications, e.g. Dijkstra's shortest path algorithm and Heapsort

Deleting from the heap

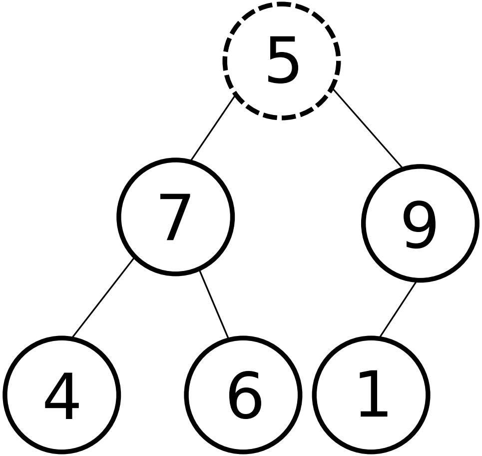
The root node is deleted and replaced with the last node

$\rightarrow$ heap balanced, but heap property violated

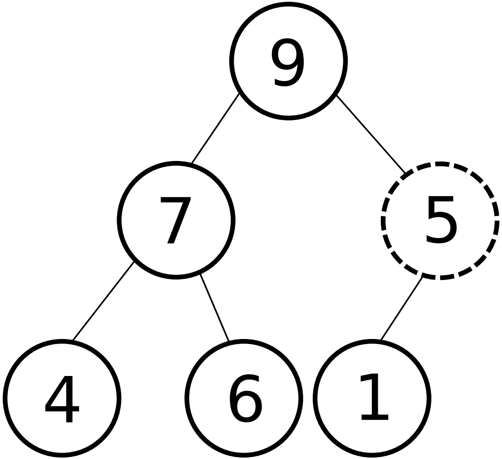
$\rightarrow$ heapify(root)



heapify



(a)

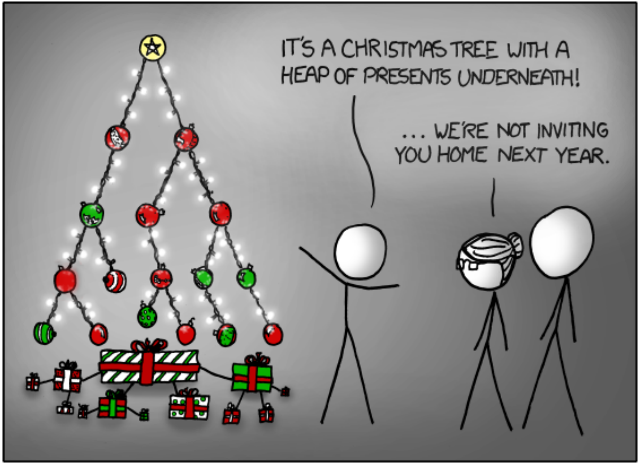


(b)

```
function H = heapify(H, n, endIndex)
  largest = 0;
  l = left(n)
  r = right(n)
  if (l <= endIndex && H(l) > H(n))
    largest = l;
  else
    largest = n;
  end
  if (r <= endIndex && H(r) > H(largest))
    largest = r;
  end
  if (largest != n)
    H = swap(H, largest, n); % pseudo-code
    H = heapify(H, largest, endIndex);
  end
end
```

Complexity of heap operations

- Insert/delete: $O(\log n)$
- Search max: $O(1)$

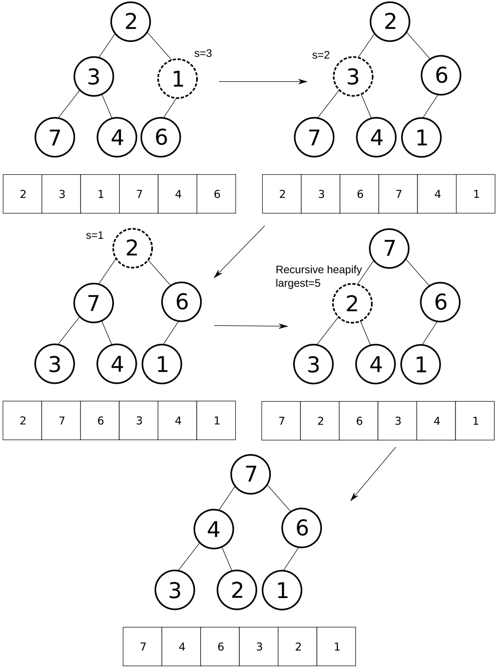


Heapsort

Step 1: turning an arbitrary array into a heap

```
function A = buildHeap(A)
  s = floor(length(A)/2);
  while (s > 0)
    A = heapify(A, s, length(A));
    s = s - 1;
  end
end
```

Let's heapsort [2 3 1 7 4 6]


s=3
s=2
s=1
Recursive heapify
largest=5
2 3 1 7 4 6
2 3 6 7 4 1
2 7 6 3 4 1
7 2 6 3 4 1
7 4 6 3 2 1

Complexity of buildHeap

```
function A = buildHeap(A)
  s = floor(length(A)/2);
  while (s > 0)
    A = heapify(A, s, length(A));
    s = s - 1;
  end
end
```

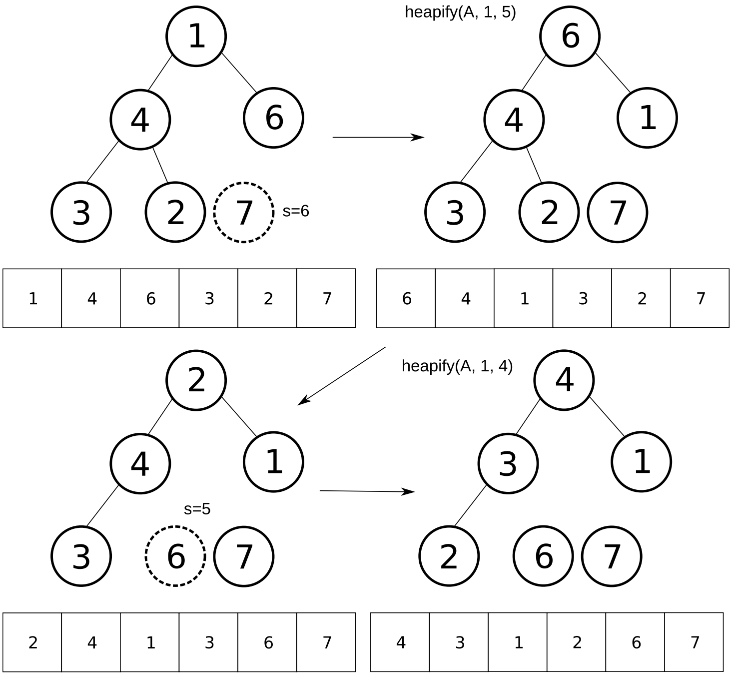
Assuming procedures (which we do not have in Matlab):

- $n/2$ iterations of while-loop
- n-node heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
- heapify with heap of height h is $O(h)$

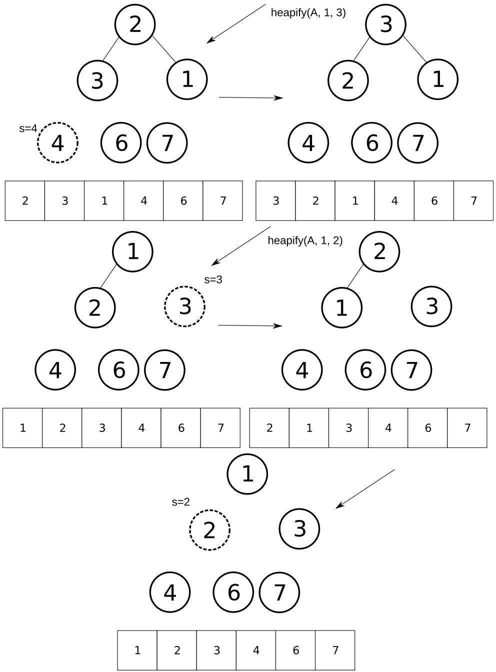
$\Rightarrow \sum_{h=0}^{\lfloor \log_2 n \rfloor} \lceil \frac{n}{2^{h+1}} \rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{h}{2^h}\right) = O(n2) = O(n)$

Full heapsort

```
function A = heapSort(A)
  s = length(A);
  % until s == 2, but this is a safer condition
  while (s > 1)
    % pseudo-code
    A = swap(A, 1, s);
    A = heapify(A, 1, s);
    s = s - 1;
  end
end
```



heapify(A, 1, 5)
s=6
1 4 6 3 2 7
6 4 1 3 2 7
heapify(A, 1, 4)
s=5
2 4 1 3 6 7
4 3 1 2 6 7


heapify(A, 1, 3)
s=4
2 3 1 4 6 7
3 2 1 4 6 7
heapify(A, 1, 2)
s=3
1 2 3 4 6 7
2 1 3 4 6 7
s=2
1 2 3 4 6 7

Complexity of heapsort

- Initial build heap: $O(n)$
- heapSort: n iterations of heapify, each $O(\log n)$
- Total: $O(n) + O(n \log n) = O(n \log n)$
- And does the sorting in place!