Programming (Econometrics)

Lecture 3: Memory organization

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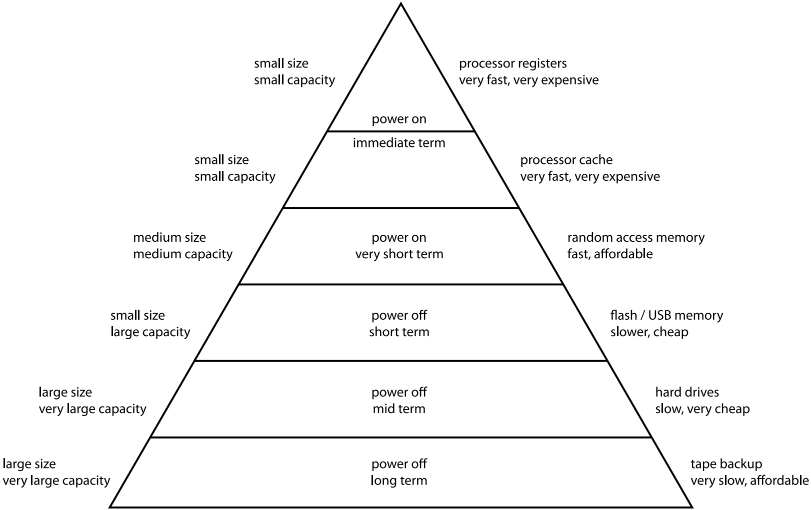
Computer Memory Hierarchy



Memory organization

- Local variables such as loop counters can possibly be stored in registers
- All larger data structures have to be allocated to the main memory
- The random access memory is linear and addressed using integers pointing out the location (e.g. 0x400345CF)
- 32 bit adrressing = max 4Gb of memory

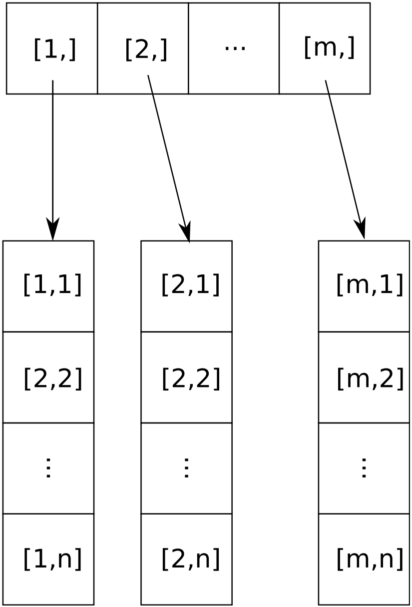
Matrices in Matlab

- Matrices are included in Matlab as a built-in data type

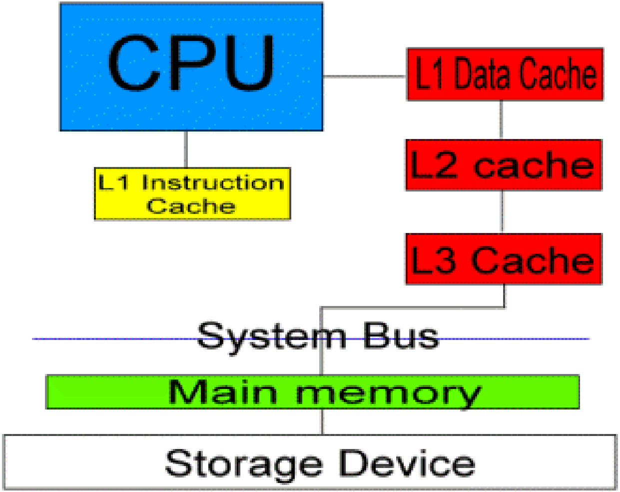
```
a = [3, 4];
b = '1';
c = a*b; % what's c now?
```

- How to represent $m \times n$ matrices?

Matrix representations: naive



CPU caches


CPU
L1 Data Cache
L1 Instruction Cache
L2 cache
L3 Cache
System Bus
Main memory
Storage Device

Matrix representations: efficient

- Memory is linear, so store the element $[a, b]$ in index $[(a-1)*n+b]$

1	2	···	n	(n+1)	(n+2)	···	(m*n)

- Row-major representation; in column-major one $[a, b]$ is in $[(b-1)*m+a]$

- In most programming languages the array indices start from 0 and the formulas are simpler

Row- and column-major representations: an example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- As row-major: [1 2 3 4 5 6]
- As column-major: [1 4 2 5 3 6]

Special matrices: sparse

- If the matrix if *sparse*, i.e. it contains only a few elements, it is more efficient to store only the non-zero elements
- E.g.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Can be represented with ([3, 4, 2], [5, 1, 1])

Special matrices: diagonal and identity

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

- Can be represented with [1, 3, 2, 7, 4]

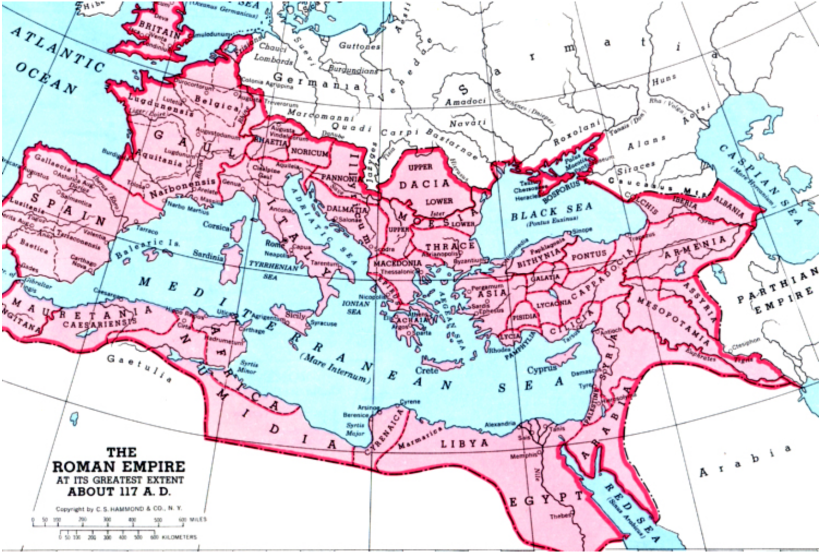
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- $= I_5$ and can be represented with a single integer 5

Matrix multiplication: naive

```
function C = multiply(A, B)
  C = zeros(rows(A), columns(B));
  for (i=1:rows(A))
    for (j=1:columns(B))
      s = 0;
      for (k=1:columns(A))
        s = s + A(i, k) * B(k, j);
      end
      C(i, j) = s;
    end
  end
end
```

- Complexity?


THE ROMAN EMPIRE
AT ITS GREATEST EXTENT
ABOUT 117 A. D.
Copyright by C. S. HAMMOND & CO., N. Y.
0 50 100 200 300 400 500 600 MILES
0 50 100 200 300 400 500 600 KILOMETERS
ATLANTIC OCEAN
BRITAIN
Germania
Sarmatia
GAUL
Belgica
Lugdunensis
Aquitania
Narbonensis
SPAIN
Lusitania
Baetica
Tarraconensis
Gallaecia
MAURETANIA CAESARIENSIS
NUMIDIA
AFRICA
Gaetulia
ITALY
RHAETIA
NORICUM
PANNONIA
DALMATIA
DACIA
UPPER
LOWER
MOESIA
THRACE
MACEDONIA
ACHAIA
EPIRUS
ASIA
BITHYNIA
PONTUS
GALATIA
CAPPADOCIA
LYCAONIA
CILICIA
PISIDIA
LYCIA
ARMENIA
ASSYRIA
MESOPOTAMIA
SYRIA
ARABIA
PALESTINE
EGYPT
LIBYA
CYRENAICA
Marmarica
IBERIA
ALBANIA
COLCHIS
BOSPORUS
BLACK SEA
(Pontus Euxinus)
CASPIAN SEA
MEDITERRANEAN SEA
(Mare Internum)
ADRIATIC SEA
TYRRHENIAN SEA
IONIAN SEA
AEGEAN SEA
RED SEA
(Sinus Arabicus)
PARTHIAN EMPIRE
Arabia
Corsica
Sardinia
Sicily
Crete
Cyprus
Rhodes
Balearic Is.
Rome
Carthage
Alexandria
Antioch
Byzantium
Athens
Sparta
Syracuse
Cyrene
Memphis
Thebes
Damascus
Tyre
Sinope
Trapezus
Ctesiphon
Euphrates
Tigris
Nile
Danube
Chauci
Lombards
Guttones
Burgundians
Suevi
Venedae
Marcomanni
Quadi
Carpi
Bastarnae
Jazyges
Amadoci
Navari
Roxolani
Alans
Siraces
Huns
Aorsi
Caucasus Mts.

Matrix multiplication: divide-and-conquer

- Assume that we are multiplying $n \times n$ matrices, where n is a power of 2

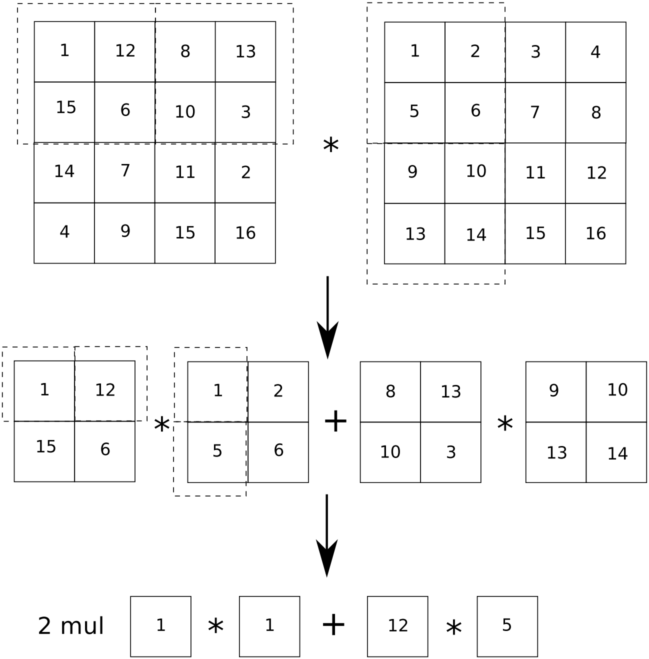
- Express $C = AB$ as

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

that comes down to computing

$$\begin{aligned} C_{1,1} &= A_{1,1}B_{1,1} + A_{1,2}B_{2,1} \\ C_{1,2} &= A_{1,1}B_{1,2} + A_{1,2}B_{1,2} \\ C_{2,1} &= A_{2,1}B_{1,1} + A_{2,2}B_{2,1} \\ C_{2,2} &= A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{aligned}$$

- Proceed recursively until you multiply matrices of max size 1×1



1
12
8
13
15
6
10
3
14
7
11
2
4
9
15
16
*
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
1
12
15
6
*
1
2
5
6
+
8
13
10
3
*
9
10
13
14
2 mul
1
*
1
+
12
*
5

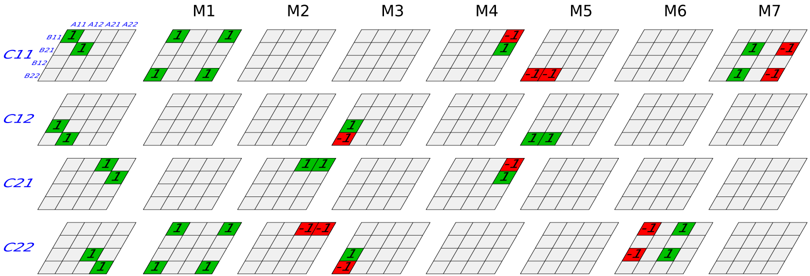
Complexity of divide-and-conquer multiplication

$$
\begin{aligned}
T(n) &= 8T(n/2) + n^2 \\
&= n^2 + 8((n/2)^2 + 8T(n/4)) \\
&= n^2 + 8((n/2)^2 + 8((n/4)^2 + 8T(n/16))) \\
&= n^2 + 2n^2 + 4n^2 + 8T(n/16)))
\end{aligned}
$$

i^{th} term in the series is $2^{i-1}n^2$

$$
\begin{aligned}
T(n) &= n^2 + 2n^2 + 4n^2 + \cdots + 2^{\log_2 n} O(1) \\
&= n^2 \sum_{i=0}^{\log_2 n} 2^i + O(n^{\log_2 2}) \\
&= n^2 \frac{2^{\log_2(n+1)} - 1}{2 - 1} + O(n) \\
&\leq n^2 O(2^{\log_2 n}) + O(n) = n^2 O(n) + O(n) \\
&= O(n^3)
\end{aligned}
$$

Strassen's idea



Now we only need to do 7 multiplications, so the complexity becomes

$$
\begin{aligned}
T(n) &= 7T(n/2) + O(n^2) \\
&= O(n^{\log_2 7}) \approx O(n^{2.81})
\end{aligned}
$$

Static data structures

- Matrices and arrays are static data structures in the sense that although accessing an arbitrary element is efficient, adding an element is not
- Example: add an element into an array

Complexity of operations with matrices and arrays

For n elements

- Add element: O(n)
- Random access: O(1)
- Delete element: O(n)

For $n \times n$ matrices:

- Multiplication: $O(n^3)$ (?)
- Inversion: as multiplication
- Determinant: $O(n^3)$ with LU decomposition