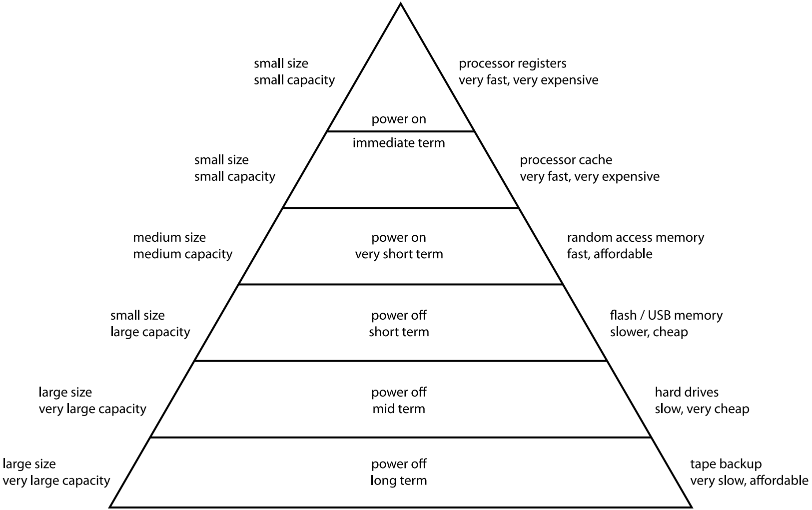
Programmeren (Ectrie)

Lecture 3: Memory organization

Tommi Tervonen

Econometric Institute, Erasmus University Rotterdam

Computer Memory Hierarchy



Memory organization

- Local variables such as loop counters can possibly be stored in registers
- All larger data structures have to be allocated to the main memory
- The random access memory is linear and addressed using integers pointing out the location (e.g. 0x400345CF)

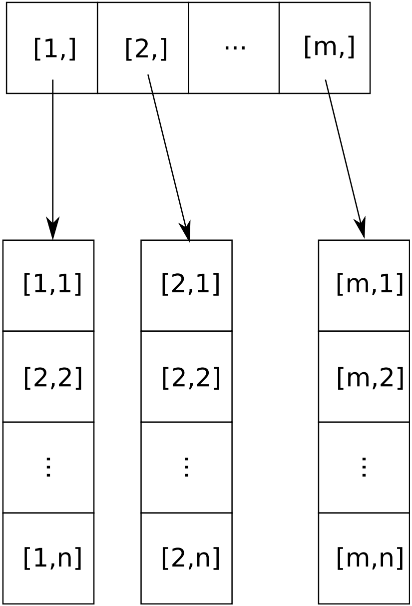
Matrices in Matlab

- Due to many algorithms being representable as matrix operations, matrices are included in Matlab as a built-in data type

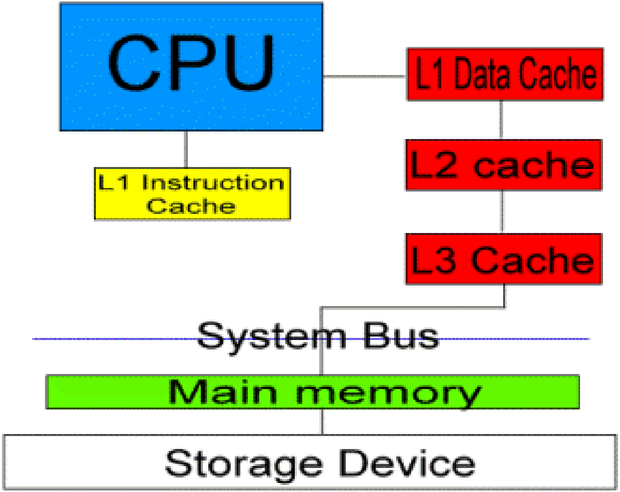
```
a = [3, 4];
b = '1';
c = a*b;
```

- now $c = ?$
- How do we represent $m \times n$ matrices?

Matrix representations: naive



CPU caches


CPU
L1 Data Cache
L1 Instruction Cache
L2 cache
L3 Cache
System Bus
Main memory
Storage Device

Matrix representations: efficient

- As the memory is linear, let's exploit that and store the element $[a, b]$ in index $[(a-1)*b+n]$

1	2	⋯	n	(n+1)	(n+2)	⋯	(m*n)

- This is called the row-major representation; in column-major one $[a, b]$ is in $[(b-1)*m+a]$
- In most programming languages the array indices start from 0 and the formulas are simpler

Row- and column-major representations: an example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- As row-major: [1 2 3 4 5 6]
- As column-major: [1 4 2 5 3 6]

Special matrices: sparse

- If the matrix if *sparse*, i.e. it contains only a few elements, it is more efficient to store only the non-zero elements
- E.g.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Can be represented with ([3, 4, 2], [5, 1, 1])

Special matrices: diagonal and identity

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

- Can be represented with [1, 3, 2, 7, 4]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- $= I_5$ and can be represented with a single integer 5

Matrix multiplication: naive

```
function C = multiply(A, B)
  C = zeros(rows(A), columns(B));
  for (i=1:rows(A))
    for (j=1:columns(B))
      s = 0;
      for (k=1:columns(A))
        s = s + A(i, k) * B(k, j);
      end
      C(i, j) = s;
    end
  end
end
```

- Complexity?

Matrix multiplication: divide-and-conquer

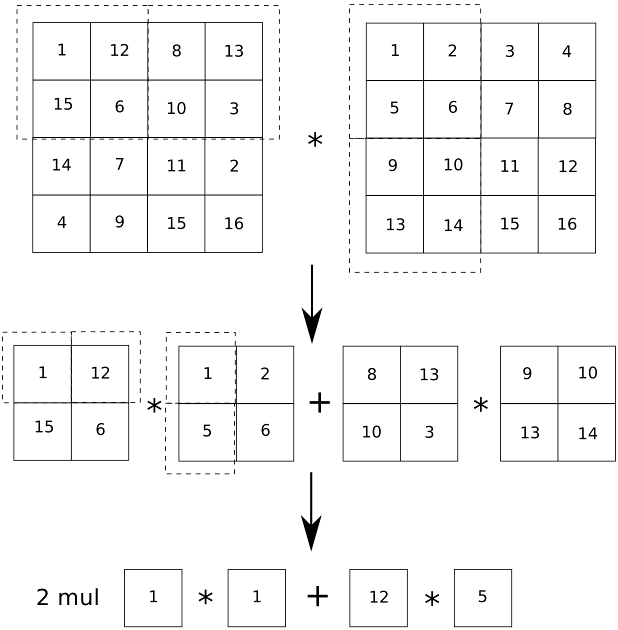
- Assume that we are multiplying $n \times n$ matrices, where n is a power of 2
- Express $C = AB$ as

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

  that comes down to computing

$$\begin{aligned} C_{1,1} &= A_{1,1}B_{1,1} + A_{1,2}B_{2,1} \\ C_{1,2} &= A_{1,1}B_{1,2} + A_{1,2}B_{1,2} \\ C_{2,1} &= A_{2,1}B_{1,1} + A_{2,2}B_{2,1} \\ C_{2,2} &= A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{aligned}$$

- Proceed recursively until you multiply matrices of max size 1×1


1 12 8 13
15 6 10 3
14 7 11 2
4 9 15 16
*
1 2 3 4
5 6 7 8
9 10 11 12
13 14 15 16
1 12
15 6
*
1 2
5 6
+
8 13
10 3
*
9 10
13 14
2 mul
1 * 1 + 12 * 5

Complexity of divide-and-conquer multiplication

$$\begin{aligned}
T(n) &= 8T(n/2) + n^2 \\
&= n^2 + 8((n/2)^2 + 8T(n/4)) \\
&= n^2 + 8((n/2)^2 + 8((n/4)^2 + 8T(n/16))) \\
&= n^2 + 2n^2 + 4n^2 + 8T(n/16)))
\end{aligned}$$

Complexity of divide-and-conquer multiplication

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i^{th} term in the series is $2^{i-1}n^2$

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i^{th} term in the series is $2^{i-1}n^2$

$$\begin{aligned} T(n) &= n^2 + 2n^2 + 4n^2 + \cdots + 2^{\log_2 n} O(1) \\ &= n^2 \sum_{i=0}^{\log_2 n} 2^i + O(n^{\log_2 2}) \\ &= n^2 \frac{2^{\log_2 (n+1)} - 1}{2-1} + O(n) \\ &\leq n^2 O(2^{\log_2 n}) + O(n) = n^2 O(n) + O(n) \\ &= O(n^3) \end{aligned}$$

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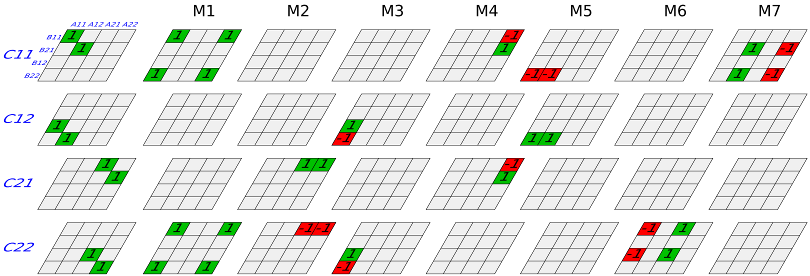
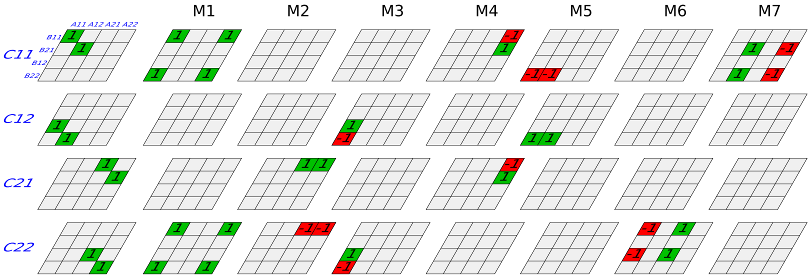
Strassen's idea



Strassen's idea



Now we only need to do 7 multiplications, so the complexity becomes

$$\begin{aligned} T(n) &= 7T(n/2) + O(n^2) \\ &= O(n^{\log_2 7}) \approx O(n^{2.81}) \end{aligned}$$

Static data structures

- Matrices and arrays are static data structures in the sense that although accessing an arbitrary element is efficient, adding an element is not
- Example: add an element into an array

Complexity of operations with matrices and arrays

For n elements

- Add element: O(n)
- Random access: O(1)
- Delete element: O(n)

Additionally, for $n \times n$ matrices:

- Multiplication: O(n^3) (?)
- Inversion: as multiplication
- Determinant: O(n^3) with LU decomposition