Programmeren (Ectrie)

Lecture 2: Computing

Tommi Tervonen

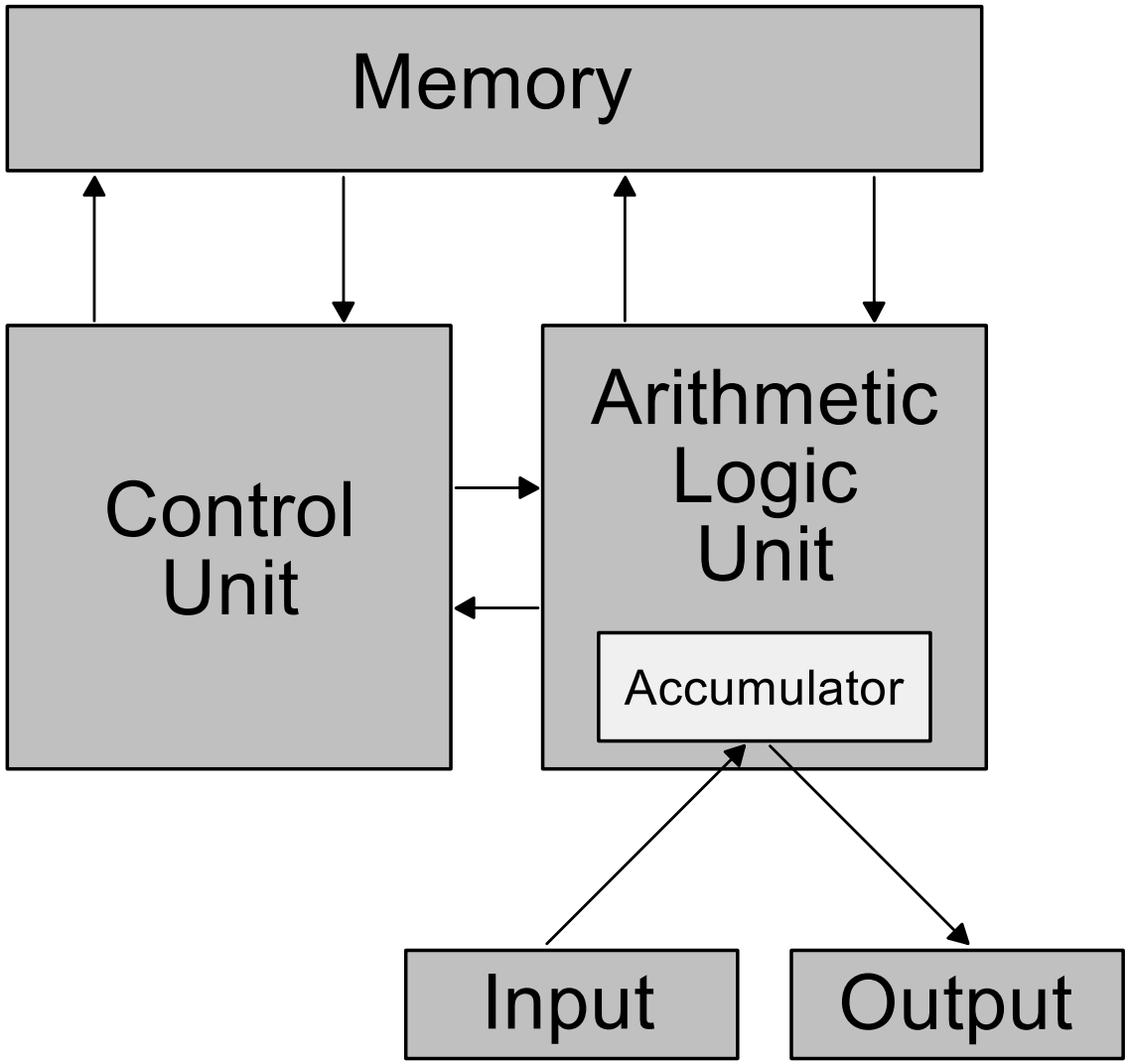
Econometric Institute, Erasmus University Rotterdam

What's the difference?

Stored-program computers

- Stored-program computers allow self-modifying code
- Enable to write, compile, and run code on the same machine
- Implement *von Neumann* architecture

von Neumann architecture



Numerical representation

- von Neumann computers have instruction set (e.g. MOV, MUL, ADD, LEA, ROL, ...) where for each instruction we have an *opcode* that is the instruction representation in binary format
- Numbers (integers and reals) are also just a set of bits
- Standard computers operate with a certain number of bits (usually 32, although new processors are 64 bit)
- We give *semantics* to the sequences of bits to represent integers, reals, characters, opcodes, ...

Floating point numbers

- Whereas integers are always of a certain range (e.g. standard 32-bit: $[-2^{31}, 2^{31} - 1]$, the limited-bit representation of real numbers causes them to be of a certain accuracy
- Reals are represented as *floating point numbers*. For a fixed base b and number of digits p as
  - Signed fraction f
  - Exponent e

$$(e, f) = f \times b^e$$

E.g. floating decimal ($b = 10$) with 8 digits can represent Plank's constant (6.6261×10^{-27}) as

$$(-26, +.66261000)$$

Floating point numbers

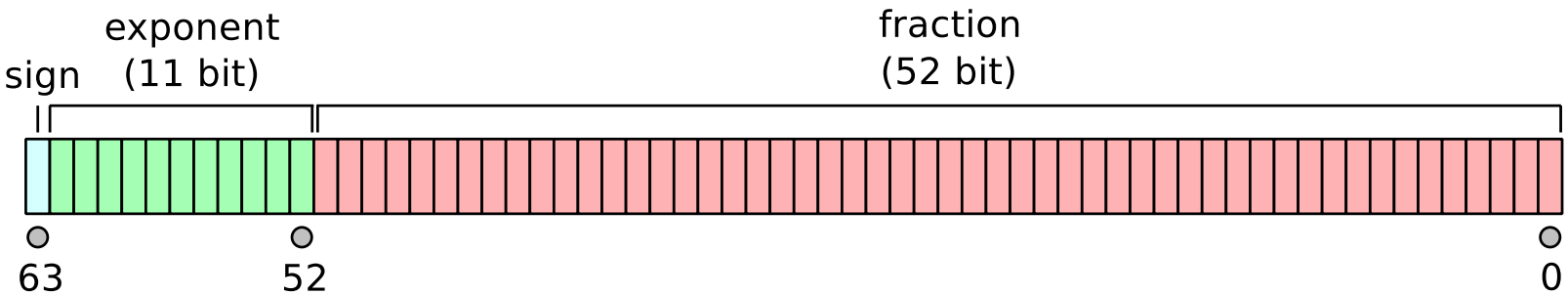
- To make floating point computation easier, we represent the numbers in a normalized format so that

$$|f| < 1$$
$$-b^p < b^p f < b^p$$

- Standard way to denote floating point numbers in exponential format in programming languages is to present the fraction followed by 'E' and the exponent, e.g. the Planck's constant would be '0.66261E-26'

IEEE 754

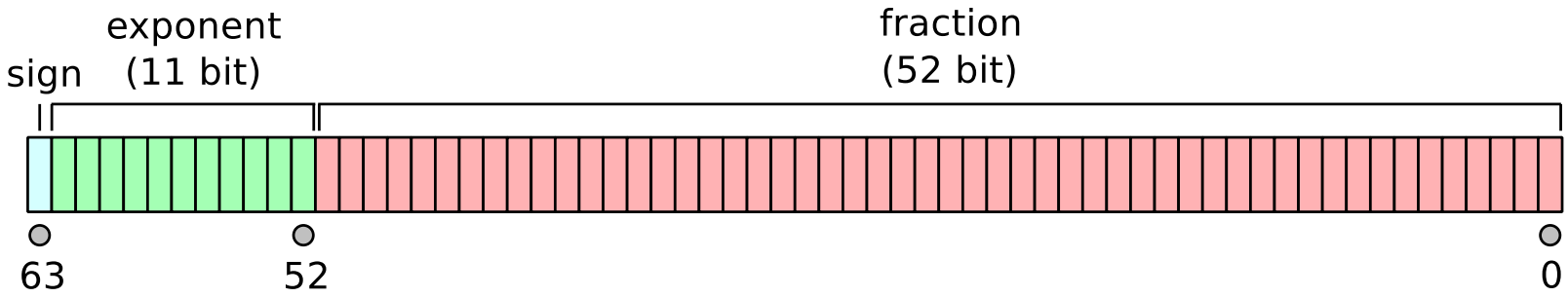
- We still need to choose b, p, and bit sizes for e and f
- Most processors support IEEE 754 double precision (64 bit) floating point standard with $b = 2$:



$$value = (-1)^{sign}(1 + \sum_{i=1}^{52} b\_i 2^{-i}) \times 2^{(e-1023)}$$

- Java's `double` and Matlab's numbers are 64 bit floats

IEEE 754



Implications:

- The decimal point is floating, and precision of the fraction is $53 \log_{10} 2 \approx 15.955$
- 2300000000000000000 ok
- 2300000000000000001 not

Problems with floating point numbers (1)

Operations on floating point numbers performed on computers are neither associative nor distributive, that is,

$$a + (b + c) \neq (a + b) + c, \text{ for many } a, b, c$$
$$a * (b + c) \neq (a * b) + (a * c), \text{ for many } a, b, c$$

when a, b, and c are floating point numbers. For example, consider

$$a = 0.42$$
$$b = -0.5$$
$$c = 0.08$$

now, when computed with IEEE 754 double-precision binary floats, we get

$$(a + b) + c = -1.3878 \times 10^{-17}$$
$$a + (b + c) = 0$$

Problems with floating point numbers (1)

- ... as not all numbers can be represented exactly with floating point numbers
- The interval between numbers that can be represented depends on the magnitude: with larger numbers the interval is larger
- So floating point numbers can be thought of representing an interval around the given value (e.g. 0.42 is $[0.42 - \epsilon_1, 0.42 + \epsilon_2]$)

Problems with floating point numbers (2)

The limited amount of bits used to store the numbers can cause under- and overflows:

$$1.2345678 + 1.7654321 = 3$$

due to the inherent imprecision of the floating point representation $\Rightarrow$ you should never compare results against an exact value, but rather see whether they are within some threshold ϵ:

$$1.2345678 + 1.7654321 - 3 \leq \epsilon$$

Problems with floating point numbers (3)

The accuracy of floating point numbers highly depends on the operations performed.

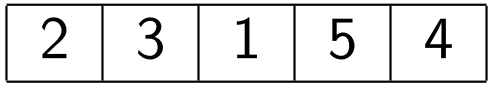
- multiplication is less precise than addition
- repeated application of addition/substraction can result in arbitrarily large errors if incorrect rounding scheme is used (though in practice it isn't)

If you need very high precision floats, Matlab isn't probably the correct language to use.

Computational complexity

- The only two resources for algorithms are computation time and memory
- Computational complexity refers to their use - how complex is the algorithm given an input of size n
- Complexity theory forms the basis for all computational sciences
- Complexity can be analyzed by counting the amount of resources used
- Example: adding together two integers 12345 and 53766

Insertion sort



- Sort in the way card game players sort their hands

Insertion sort

2	3	1	5	4

- Sort in the way card game players sort their hands

```
function insertionSort(a)
   for j=2:length(a)
      key = a(j);
      i = j-1;
      while i > 0 && a(i) > key
         a(i+1) = a(i);
         i = i-1;
      end
   a(i+1) = key
   end
end
```

Insertion sort: example

```
1  function insertionSort(a)
2    for j=2:length(a)
3      key = a(j);
4      i = j-1;
5      while i > 0 && a(i) > key
6        a(i+1) = a(i);
7        i = i-1;
8      end
9    a(i+1) = key
10   end
11 end
```

2	3	1	5	4	starting array
2	**3**	1	5	4	j=2
1	2	3	5	4	j=3
1	2	3	**5**	4	j=4
1	2	3	**4**	5	j=5

Insertion sort: analysis

Assumptions:

- We are computing with a single-processor random access machine
- No parallel processing
- Instructions are processsed sequentially
- The machine has unlimited memory

Insertion sort: analysis

- **Memory**: a constant amount of additional memory (i.e. for the temporary variables) is used - insertion sort does the sorting *in place*
- **Running time**: count the amount of primitive operations performed
  - Primitive operations = arithmetic operations, comparisons, assignments, etc
  - Exact number of CPU cycles / operation depends on compiler and hardware
  - Analyze on more abstract level by counting the amount *computation steps*

```
1  function insertionSort(a)
2     for j=2:length(a)
3        key = a(j);
4        i = j-1;
5        while i > 0 && a(i) > key
6           a(i+1) = a(i);
7           i = i-1;
8        end
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11 end
```

- Amount of times each line is executed
- c_i: the cost of executing line i
- t_j the amount of times the while loop test on line 5 is executed

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1  function insertionSort(a)
2     for j=2:length(a)
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4        i = j-1;
5        while i > 0 && a(i) > key
6           a(i+1) = a(i);
7           i = i-1;
8        end
9     a(i+1) = key
10    end
11 end
```

Line	2	3	4	5	6	7	8
Cost	c_2	c_3	c_4	c_5	c_6	c_7	c_8
Times	n	$n-1$	$n-1$	$\sum_{j=2}^{n} t_j$	$\sum_{j=2}^{n} (t_j - 1)$	$\sum_{j=2}^{n} (t_j - 1)$	$n-1$

Line	2	3	4	5	6	7	8
Cost	c_2	c_3	c_4	c_5	c_6	c_7	c_8
Times	n	$n-1$	$n-1$	$\sum_{j=2}^{n} t_j$	$\sum_{j=2}^{n}(t_j-1)$	$\sum_{j=2}^{n}(t_j-1)$	$n-1$

$$
\begin{aligned}
T(n) = & c_2 n + c_3(n-1) + c_4(n-1) \\
& + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n}(t_j-1) \\
& + c_7 \sum_{j=2}^{n}(t_j-1) + c_8(n-1)
\end{aligned}
$$

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1  function insertionSort(a)
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8      end
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10   end
11 end
```

- The running time depends on size of the input n *and* times the inner loop is executed t_j
- $a(i) \leq a(j)\ \forall i < j,\ i,j \in \{1,\ldots,n\} \Rightarrow t_j = 1\ \forall j \in \{1,\ldots,n\}$

Best-case running time

$$
\begin{aligned}
T(n) = & c_2 n + c_3(n-1) + c_4(n-1) \\
& + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) \\
& + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)
\end{aligned}
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if $a(i) \leq a(j)\ \forall i < j,\ i, j \in \{1, \ldots, n\} \Rightarrow t_j = 1\ \forall j \in \{1, \ldots, n\}$

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if $a(i) \leq a(j)\ \forall i < j,\ i,j \in \{1,\ldots,n\} \Rightarrow t_j = 1\ \forall j \in \{1,\ldots,n\}$

$$
\begin{aligned}
\Rightarrow T(n) &= c_2 n + c_3(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\
&= (c_2 + c_3 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)
\end{aligned}
$$

Best-case running time

$$\begin{aligned} T(n) =& c_2 n + c_3(n-1) + c_4(n-1) \\ &+ c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) \\ &+ c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) \end{aligned}$$

if $a(i) \leq a(j)\ \forall i < j,\ i,j \in \{1,\ldots,n\} \Rightarrow t_j = 1\ \forall j \in \{1,\ldots,n\}$

$$\begin{aligned} \Rightarrow T(n) &= c_2 n + c_3(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\ &= (c_2 + c_3 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) \end{aligned}$$

replace $c_2 + c_3 + c_4 + c_5 + c_8 = a$ and $c_2 + c_4 + c_5 + c_8 = b$

Best-case running time

$$\begin{aligned} T(n) =& c_2 n + c_3(n-1) + c_4(n-1) \\ &+ c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) \\ &+ c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) \end{aligned}$$

if $a(i) \leq a(j)\ \forall i < j,\ i,j \in \{1,\ldots,n\} \Rightarrow t_j = 1\ \forall j \in \{1,\ldots,n\}$

$$\begin{aligned} \Rightarrow T(n) &= c_2 n + c_3(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\ &= (c_2 + c_3 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) \end{aligned}$$

replace $c_2 + c_3 + c_4 + c_5 + c_8 = a$ and $c_2 + c_4 + c_5 + c_8 = b$

$$\Rightarrow T(n) = an + b$$

```
1  function insertionSort(a)
2     for j=2:length(a)
3        key = a(j);
4        i = j-1;
5        while i > 0 && a(i) > key
6           a(i+1) = a(i);
7           i = i-1;
8        end
9     a(i+1) = key
10    end
11 end
```

If $a(i) > a(j)\ \forall i < j,\ i,j \in \{1,\ldots,n\}$
$\Rightarrow$ in every iteration of the while loop the current element a(i) must be compared with each of the elements in the already sorted subarray $a(1),\ldots,a(i-1)$, so $t_j = j\ \forall j \in \{2,\ldots,n\}$

$$T(n) = c_2 n + c_3(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} j$$
$$+ c_6 \sum_{j=2}^{n} (j-1) + c_7 \sum_{j=2}^{n} (j-1) + c_8(n-1)$$

note that

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \text{ and } \sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

$$T(n) = c_2 n + c_3(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} j$$
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note that

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \text{ and } \sum_{j=2}^{n}(j-1) = \frac{n(n-1)}{2}$$

$$\Rightarrow T(n) = c_2 n + c_3(n-1) + c_4(n-1) + c_5(\frac{n(n+1)}{2} - 1)$$
$$+ (c_6 + c_7)(\frac{n(n-1)}{2}) + c_8(n-1)$$
$$= (\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2})n^2$$
$$+ (c_2 + c_3 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8)n$$
$$- (c_3 + c_4 + c_5 + c_8)$$

Worst-case running time

$$
\begin{aligned}
T(n) = &\left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 \\
&+ \left(c_2 + c_3 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n \\
&- \left(c_3 + c_4 + c_5 + c_8\right)
\end{aligned}
$$

replace sets of c_i's with constants a, b, and c

Worst-case running time

$$
\begin{aligned}
T(n) = &\left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 \\
&+ \left(c_2 + c_3 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n \\
&- \left(c_3 + c_4 + c_5 + c_8\right)
\end{aligned}
$$

replace sets of c_i's with constants a, b, and c

$$\Rightarrow T(n) = an^2 + bn + c$$

Analysis: conclusions

Insertion sort

- Sorts in place - requires constant amount of memory not dependent on the input size
- Has linear best-case complexity
- Has quadratic worst-case complexity

Worst-case analyses

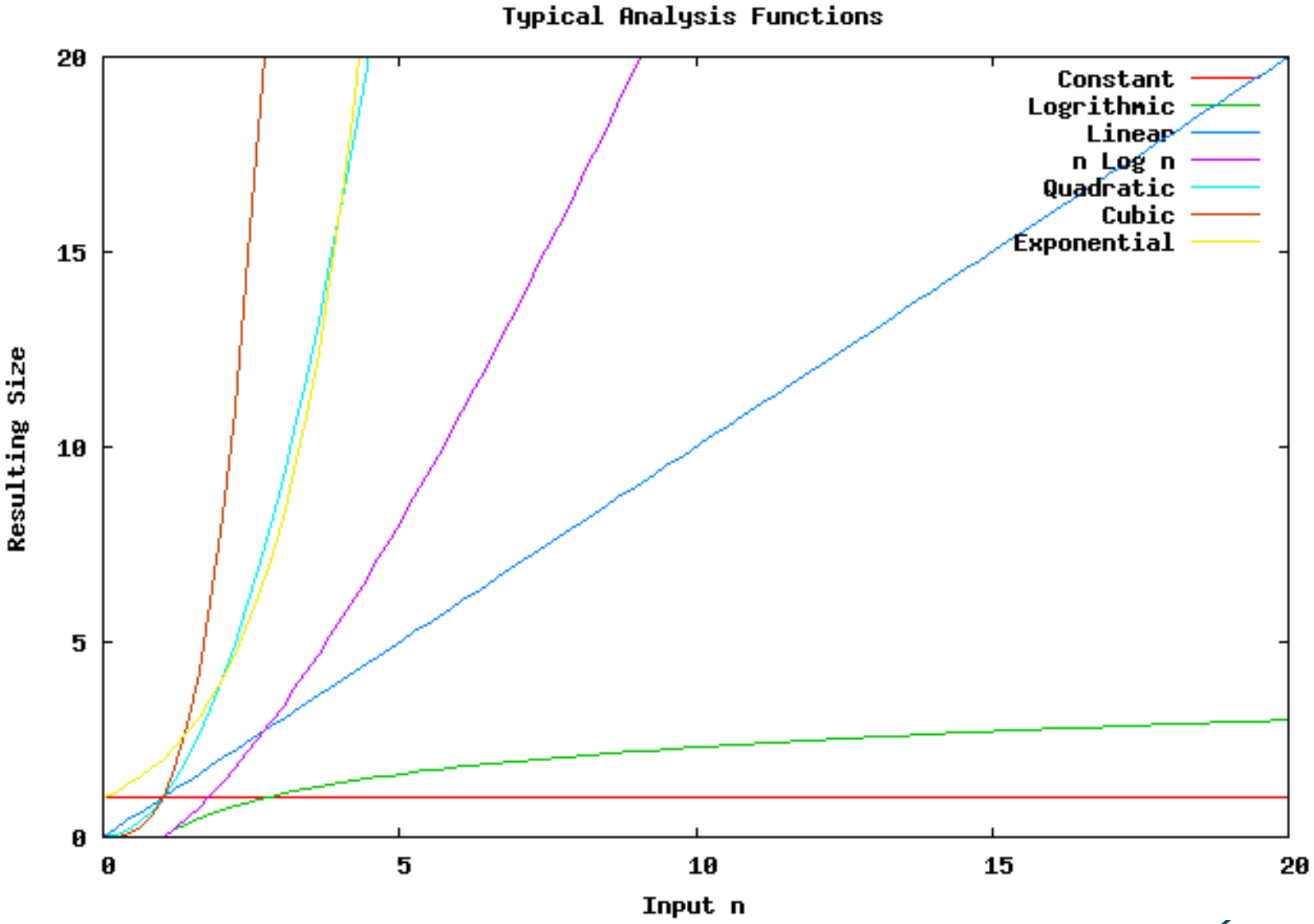
Usually we are interested *only* in the worst-case complexity, as

- It gives us an upper-bound on how bad the algorithm can perform
- Worst-case occurs fairly often with some algorithms
- Worst-case can occur with extremely high probability when input is from real-life processes (e.g. sorting customers)
- Algorithms are executed often, so worst case happens almost surely sometime

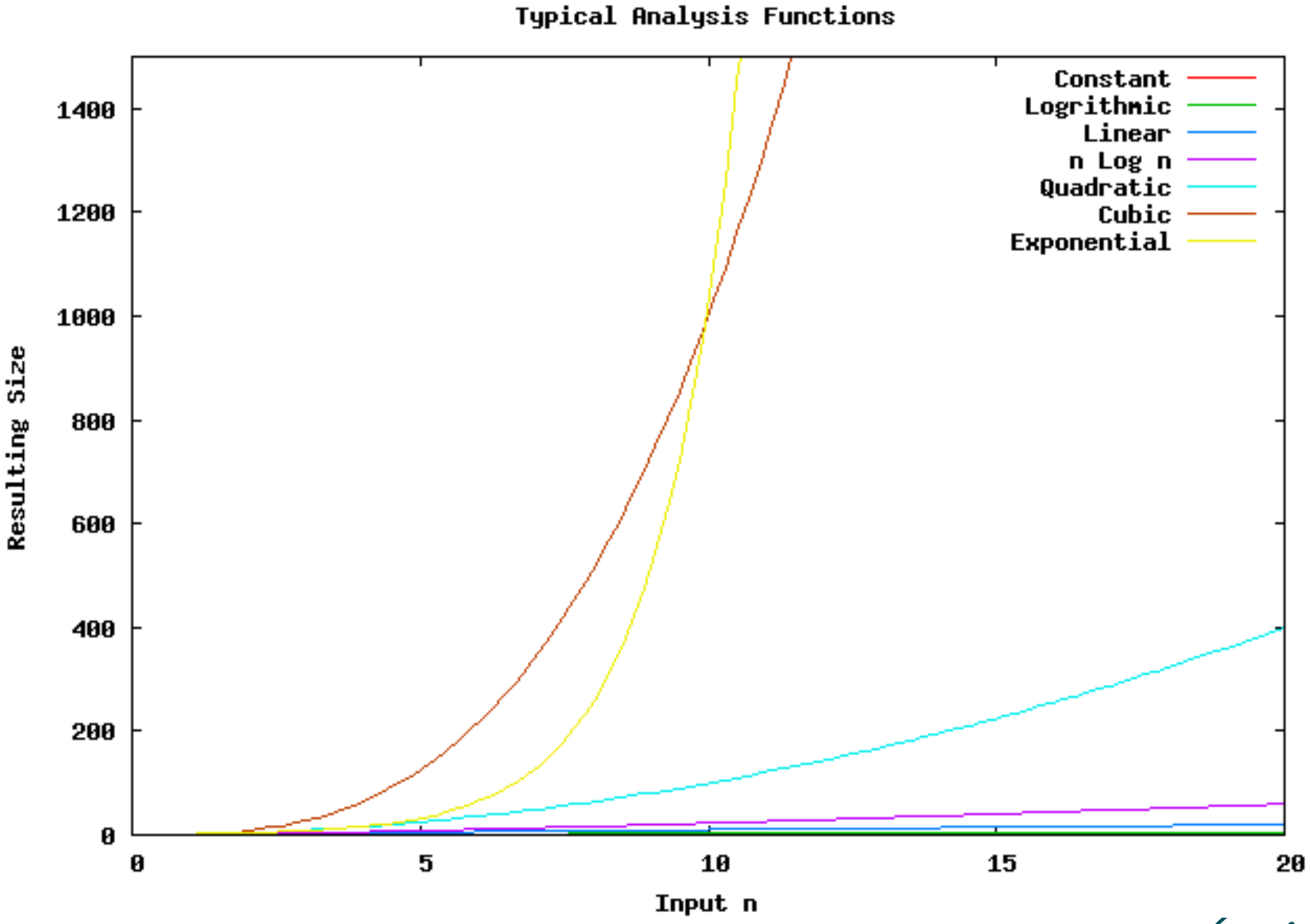
Running times with a computer processing 10^9 ops/s

f(n)	10	100	1000	10^4	10^5	10^6
n	10^{-8}s	10^{-7}s	10^{-6}s	10^{-5}s	10^{-4}s	10^{-3}s
$n \log n$	10^{-8}s	2.4×10^{-8}s	2.0×10^{-6}s	3.5×10^{-4}s	0.1s	56s
n^2	10^{-7}s	10^{-5}s	10^{-3}s	0.1s	10s	17min
n^3	10^{-6}s	10^{-3}s	1s	17min	12d	32y
2^n	10^{-6}s	4.0×10^{13}y	3.3×10^{284}y			
$n!$	3.6×10^{-3}s	3.0×10^{141}y				

Growth of functions

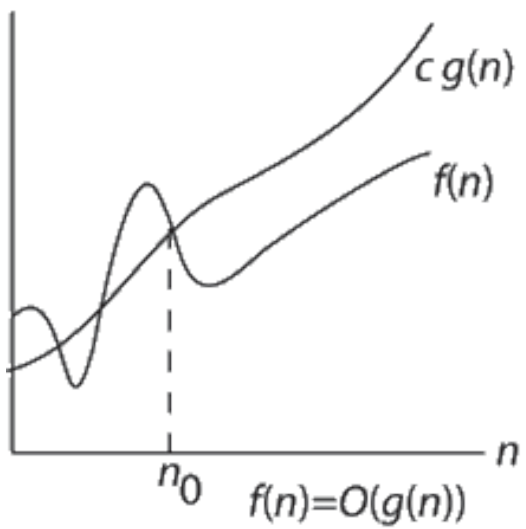


Growth of functions

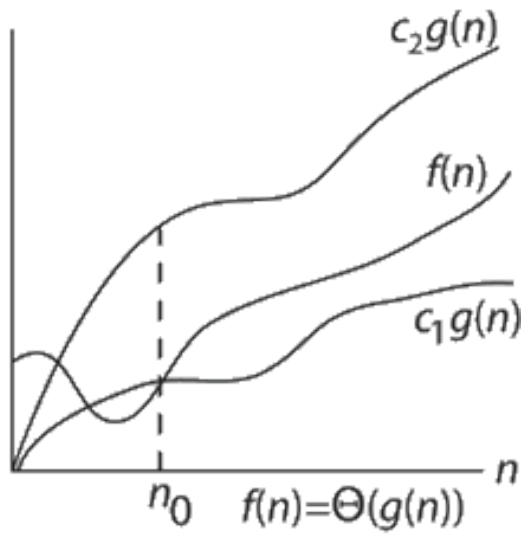


Asymptotic complexity

- Exact analysis as we did before (with c_i's) is not meaningful - only *asymptotic* complexity matters
- Given an input size $n > n_0$, where n_0 is some constant value, how fast does the running time grow?
- The asymptotic behaviour of a function depends only on the highest order term, and not at all of the constants (the c's)



(a)



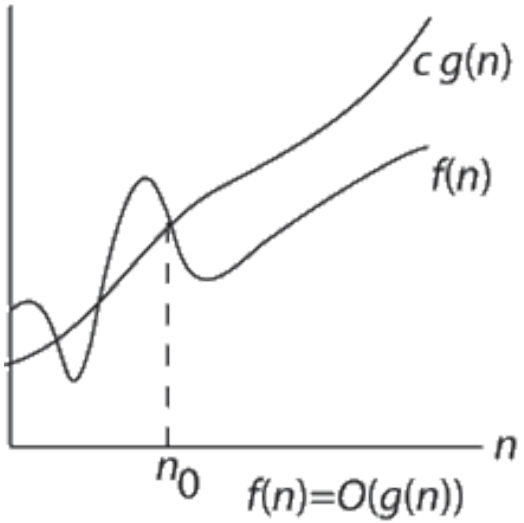
(b)

big-O notation

- For asymptotic worst-case complexity, we use the big-O notation. Given a function $g(n)$, the set of functions

$$O(g(n)) = \{f(n) : \exists c > 0, n_0 > 0 : \ 0 \leq f(n) \leq cg(n) \quad \forall n \geq n_0\}$$

are asymptotically O-equivalent.



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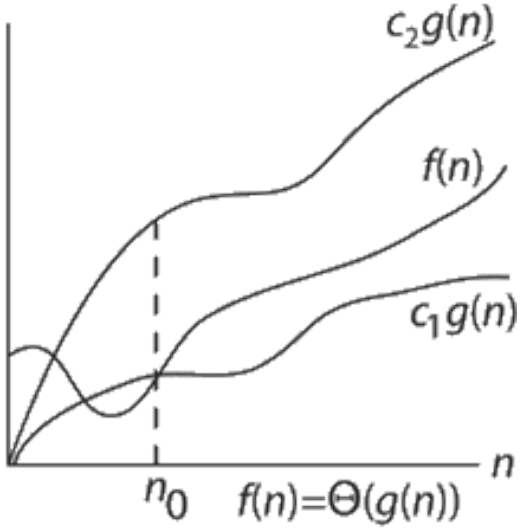

(b)

O(g(n)) is the asymptotical upper bound, that is not necessary tight

big-O

- For example, our previous quadratic complexity
  $an^2 + bn + c \in O(n^2)$, also
  $3n^2 \in O(n^2)$ and
  $\mathrm{m}n^2 + nlogn \in O(n^2)$
- Common complexity classes
  - O(1)
  - O(n)
  - O(n log n)
  - $O(n^2)$
  - $O(n^3)$
  - $O(2^n)$
  - O(n!)

About hardness

- Any problem with a known algorithm for solving it in polynomial time (O(g(n)) where g(n) is a polynomial) is called *tractable*: the algorithm grows sufficiently slow to be of possibly pratical use
- Unfortunately many practical problems in management science are intractable
- The most significant problem in mathematics: P=NP?